Assignment: Newton's Fundamental Formula

Numerical Analysis Course

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1 Question

Prove Newton's Fundamental Formula.
Newton's Fundamental Formula: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be any function and let $x_0, ...x_n$ be distinct real numbers. Then for any $x \in \mathbb{R}$

$$f(x) = p_n(x) + R_n(x)$$

where p_n is the unique polynomial of degree at most n that interpolates f at $x_0, ...x_n$ and the remainder term $R_n(x)$ is given by

$$R_n(x) = \left[\prod_{i=0}^{n}(x - x_i)\right] f[x, x_0, x_1, .., x_n]$$

2 Answer

Induction Hypothesis: Let the statement in the Question be true when the number of points are $\leq n$.

- **Base Case:** For 1 point x_0 our interpolating polynomial of degree ≤ 1 is

$$p_0(x) = f(x_0)$$

So,

$$RHS = f(x_0)+(x-x_0)\cdot f[x, x_0] = f(x_0)+f(x)-f(x_0) = f(x) = LHS$$

And

$$R_0(x) = (x - x_0) \cdot f[x, x_0])$$

And we know that $p_0(x)$ is unique in this case and thus this shows that the Induction Hypothesis holds for 1 point.

- **Assumption:** For $x_0, x_1, ..., x_{n-1}$ we have by the Induction Hypothesis that:

  $$f(x) = p_{n-1}(x) + R_{n-1}(x)$$

  where,

  $$p_{n-1} = f(x_0)+(x-x_0)f[x_0, x_1]+...+(x-x_0)(x-x_1)..(x-x_{n-2})f[x_0, x_1, .., x_{n-1}]$$

  is the unique polynomial (found by Newton's Divided Difference Formula) of degree at most n that interpolates f at $x_0, ...x_{n-1}$ and the remainder term $R_{n-1}(x)$ is given by

  $$R_{n-1}(x) = \left[\prod_{i=0}^{n-1}(x - x_i)\right] f[x, x_0, x_1, .., x_{n-1}]$$

- **Induction Step:** For $x_0, x_1, ..., x_{n-1}, x_n$ we can write the interpolating polynomial $p_n(x)$ using Newton's Divided Difference Interpolation Formula:

  $$p_n(x) = f(x_0)+(x-x_0)f[x_0, x_1]+...+(x-x_0)(x-x_1)..(x-x_{n-2})f[x_0, x_1, .., x_{n-1}] + (x - x_0)(x - x_1)..(x - x_{n-1})f[x_0, x_1, .., x_{n-1}, x_n]$$

  So, Now we know that this interpolating polynomial is unique because Langrange's Interpolation Polynomial is the unique polynomial of degree at most n that passes through n+1 points, but from Newton's Divided Formula we generated a Polynomial which passes through all the n+1 points and is of degree at most n. Hence, the polynomial must be the Langrange's Polynomial (written in a compact way). Thus, It is unique. (Refer Appendix for the Proof of this polynomial passing through all the n+1 points)
  Now, we can write,

  $$p_n(x) = p_{n-1}(x) + (x - x_0)..(x - x_{n-1})f[x_0, x_1, ..x_n]$$
  $$= f(x) - R_{n-1}(x) + (x - x_0)..(x - x_{n-1})f[x_0, x_1, .., x_n]$$
  $$= f(x) - \left[\prod_{i=0}^{n-1}(x - x_i)\right] \cdot \left(\frac{f[x, x_0, x_1, .., x_{n-1}] - f[x_0, x_1, .., x_n]}{x - x_n}\right) \cdot (x - x_n)$$
  $$= f(x) - \left[\prod_{i=0}^{n}(x - x_i)\right] \cdot f[x, x_0, ..., x_n] \implies f(x) = p_n(x) + R_n(x)$$

where,

$$R_n(x) = \left[\prod_{i=0}^{n}(x - x_i)\right] \cdot f[x, x_0, ..., x_n]$$

Thus, this is a contradiction to the Induction Hypothesis that it holds for $\leq n$ points cause we have just shown it holds for $n+1$ points too. So, Newton's Fundamental Formula holds for all $n \in \mathbb{N}$ number of points.

3 Appendix

Newton's Divided Difference Interpolation Formula Proof:

- **Induction Hypothesis:** For $x_1, x_2, ..., x_{n-1}, x_n$ ($\leq n$ points) we can write:

  $$(x_n - x_1)(x_n - x_2)..(x_n - x_{n-1})f[x_1, x_2, .., x_{n-1}, x_n] = f(x_n) - P_{n-2}(x_n)$$

  where, $P_{n-2}(x)$ is the unique interpolating polynomial of degree atmost $n-2$ passing through $x_1, x_2, ..., x_{n-1}$

- **Base Case:** For x_0, x_1 :

  $$f[x_1, x_2](x_2 - x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_2 - x_1) = f(x_2) - f(x_1) = f(x_2) - P_0(x_2)$$

  where, $P_0(x) = f(x_1)$ is the unique interpolating polynomial of degree 0 passing through $(x_1, f(x_1))$

- **Assumption:** For n points assume the Induction Hypothesis Statement

- **Induction Step:** For n+1 points. Let P_{n-1} be the unique Interpolating polynomial of degree atmost n-1 passing through $x_1, ..., x_n$

$$\begin{aligned}
&f[x_1, x_2, .., x_{n+1}](x_{n+1} - x_1)...(x_{n+1} - x_n) \\
&= \frac{f[x_2, .., x_{n+1}] - f[x_1, x_2, .., x_n]}{x_{n+1} - x_1}(x_{n+1} - x_1)...(x_{n+1} - x_n) \\
&= f[x_2, .., x_{n+1}](x_{n+1} - x_2)...(x_{n+1} - x_n) - f[x_1, x_2, .., x_n](x_{n+1} - x_2)...(x_{n+1} - x_n) \\
&= (f(x_{n+1}) - Q(x_{n+1})) - f[x_1, x_2, .., x_n](x_{n+1} - x_2)...(x_{n+1} - x_n) \\
&= f(x_{n+1}) - (Q(x_{n+1}) + f[x_1, x_2, .., x_n](x_{n+1} - x_2)...(x_{n+1} - x_n))
\end{aligned}$$

where, $Q(x)$ is the unique polynomial passing through $x_2, ..., x_{n+1}$ which we replaced there in place of a similar looking object of our Induction Hypothesis because of it being n points and thereby comes under our Induction Assumption.
We can now see the last bracket in the last equality step is nothing but $P_{n-1}(x)$. Since both sides are atmost n-1 degree polynomial passing through $x_1, ..., x_n$
And so,

$$f[x_1, ..., x_{n+1}](x_{n+1} - x_1)..(x_{n+1} - x_n) = f(x_{n+1}) - P_{n-1}(x_{n+1})$$

which means that our Induction Hypothesis also holds for n+1 points and therefore it holds for all $n \in \mathbb{N}$

So,

$$p_n(x) = f(x_0)+(x-x_0)f[x_0, x_1]+...+(x-x_0)(x-x_1)..(x-x_{n-2})f[x_0, x_1, .., x_{n-1}] \\ + (x - x_0)(x - x_1)..(x - x_{n-1})f[x_0, x_1, .., x_{n-1}, x_n]$$

is indeed the interpolating polynomial through $x_0, ..., x_n$, of degree atmost n and it passes through all the points. For x_n we just showed that the last term is $f(x_n) - p_{n-1}(x_n)$ where $p_{n-1}(x)$ is the unique interpolating polynomial through $x_0, ..., x_{n-1}$ and cancels the part $f(x_0) + (x - x_0)f[x_0, x_1] + ... + (x - x_0)(x - x_1)..(x - x_{n-2})f[x_0, x_1, .., x_{n-1}]$ leaving behind $f(x_n)$. Hence it passes through $(x_n, f(x_n))$ and $p_n(x)$ passes through all the other points is clearly visible.